

External tensor product of categories of perverse sheaves

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Abstract

Under some assumptions we prove that the Deligne tensor product of categories of constructible perverse sheaves on pseudomanifolds X and Y is the category of constructible perverse sheaves on $X \times Y$. The Deligne external tensor product functor is identified with the geometrical external tensor product.

1. Introduction

The aim of this article is to show that the geometrical external product of perverse sheaves is a concrete realisation of their abstract external tensor product. In more detail, there is a functor \boxtimes , which assigns to a pair of perverse sheaves F on a space X and G on a space Y their geometrical external tensor product $F \boxtimes G$, which is a perverse sheaf on the product of spaces $X \times Y$. We claim that under certain assumptions the functor \boxtimes makes the category of perverse sheaves on $X \times Y$ into Deligne's tensor product of abelian categories of perverse sheaves on X and on Y . This statement gives a little bit of support to the attempts of finding triangulated Hopf category analogs for quantum groups [7, 8].

Let us describe the objects we deal with. Unless otherwise specified, all topological spaces are assumed locally compact, locally completely paracompact, locally contractible and of finite cohomological dimension over \mathbb{C} . A stratified space for us will mean a compactifiable topological stratified pseudomanifold X with a stratification \mathcal{X} – a locally closed partition of X . In particular, X can be compact or a complex algebraic variety with algebraic strata. By definition of a pseudomanifold there exists a filtration by closed subspaces

$$\mathcal{F}_X : \quad X = X_n \supset X_{n-1} \supset \cdots \supset X_1 \supset X_0 = \emptyset,$$

such that the $S_i = X_i - X_{i-1}$ are topological manifolds, topologically disjoint union of strata and the conditions of [6, Definition I.1.1] hold.

Sheaves will be the sheaves of \mathbb{C} -vector spaces. Following Borel [3, V.3.3] we say that a complex of sheaves K is \mathcal{X} -cohomologically locally constant if $H^\bullet(K)$ are locally constant on each stratum. We say that K is \mathcal{X} -cohomologically constructible if it is \mathcal{X} -cohomologically locally constant and the stalks of $H^\bullet(K)$ are finite dimensional. The full subcategory of the bounded derived category $D^b(X)$ consisting of \mathcal{X} -cohomologically locally constant complexes is denoted $D_{\mathcal{X}}^b(X)$, its subcategory consisting of \mathcal{X} -cohomologically constructible complexes is denoted $D_{\mathcal{X}}^{b,c}(X)$.

Assume that (X, \mathcal{X}) is equipped with a function $p : \mathcal{X} \rightarrow \mathbb{Z}$, the perversity, satisfying the condition

$$p(S) \geq p(T) \text{ if } S \subset \overline{T}.$$

Beilinson, Bernstein and Deligne associate a t -structure on $D_{\mathcal{X}}^{b,c}(X)$ with the perversity: the full subcategory ${}^p D_{\mathcal{X}}^{\leq 0}(X)$ (resp. ${}^p D_{\mathcal{X}}^{\geq 0}(X)$) formed by complexes $K \in D_{\mathcal{X}}^{b,c}(X)$ such that for each stratum $i_S : S \hookrightarrow X$ the following holds: $H^m i_S^* K = 0$ for $m > p(S)$ (resp. $H^m i_S^! K = 0$ for $m < p(S)$). The complexes that satisfy the both conditions are called perverse sheaves. The category of perverse sheaves, the heart, is denoted

$$\text{Perv}(X) = \text{Perv}(X, \mathcal{X}, p) = {}^p D_{\mathcal{X}}^0(X) = {}^p D_{\mathcal{X}}^{\leq 0}(X) \cap {}^p D_{\mathcal{X}}^{\geq 0}(X).$$

The external tensor product functor

$$\boxtimes : D_{\mathcal{X}}^{b,c}(X) \times D_{\mathcal{Y}}^{b,c}(Y) \rightarrow D_{\mathcal{X} \times \mathcal{Y}}^{b,c}(X \times Y)$$

is defined on $K \in D_{\mathcal{X}}^{b,c}(X)$, $M \in D_{\mathcal{Y}}^{b,c}(Y)$ as

$$K \boxtimes M = (\text{pr}_X^* K) \overset{L}{\otimes}_{\mathbb{C}} (\text{pr}_Y^* M) = (\text{pr}_X^* K) \otimes_{\mathbb{C}} (\text{pr}_Y^* M),$$

where $\text{pr}_X : X \times Y \rightarrow X$, $\text{pr}_Y : X \times Y \rightarrow Y$ are the projections. We will abuse the notation denoting the derived functors in the same way as for sheaves, the prefixes R and L will be often omitted. Our main result is the following.

Theorem. *The restriction of the external tensor product functor to perverse sheaves gives a functor*

$$\boxtimes : \text{Perv}(X, \mathcal{X}, p) \times \text{Perv}(Y, \mathcal{Y}, q) \rightarrow \text{Perv}(X \times Y, \mathcal{X} \times \mathcal{Y}, p \dot{+} q),$$

where the perversity $p \dot{+} q$ is given by $(p \dot{+} q)(S \times T) = p(S) + q(T)$ for $S \in \mathcal{X}$, $T \in \mathcal{Y}$. This functor makes the target category into the Deligne tensor product of abelian \mathbb{C} -linear categories $\text{Perv}(X, \mathcal{X}, p)$ and $\text{Perv}(Y, \mathcal{Y}, q)$.

Recall that the tensor product of abelian categories is introduced by Deligne in [5].

The rest of the paper will be devoted to the proof of this theorem. In Section 2 we prove some preliminary and technical results, the main of which is the isomorphism

$$R\text{Hom}(A, C) \boxtimes R\text{Hom}(B, D) \xrightarrow{\sim} R\text{Hom}(A \boxtimes B, C \boxtimes D)$$

for cohomologically constructible complexes (Corollary 2.7). Using it we show that \boxtimes is t -exact, and that \boxtimes restricts to perverse sheaves (Corollary 2.11). In Section 3 we study simple perverse sheaves. The relationship between $R\text{Hom}^\bullet$ and \boxtimes is considered in Section 4. We reformulate our main theorem in Section 5 and prove it in a sequence of lemmas. Appendix A contains a list of useful formulas.

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2. Preliminary results

2.1 Lemma (cf. [3] 10.23(2)). *Let Z be a locally compact, locally completely paracompact topological space of finite cohomological dimension. Let $A, B, C \in D^b(Z)$. Then there is a natural morphism of functors*

$$\nu : R\underline{\mathrm{Hom}}(A, B) \otimes C \rightarrow R\underline{\mathrm{Hom}}(A, B \otimes C).$$

Proof. Let us begin with sheaves A, B, C on Z . For an open $U \subset Z$ there is a mapping

$$\begin{aligned} \mathrm{Hom}_{Sh(U)}(A|_U, B|_U) \times C(U) &\rightarrow \mathrm{Hom}_{Sh(U)}(A|_U, (B \otimes C)|_U), \\ (f, h) &\mapsto [A(V) \rightarrow B(V) \otimes C(V) \rightarrow (B \otimes C)(V)], \\ a &\mapsto f(V)(a) \otimes h|_V \end{aligned}$$

where $V \subset U$ is an arbitrary open subset. The above mapping factorises as follows

$$\begin{aligned} \mathrm{Hom}_{Sh(U)}(A|_U, B|_U) \times C(U) &\rightarrow \underline{\mathrm{Hom}}(A, B)(U) \otimes_{\mathbb{C}} C(U) \rightarrow \\ &\rightarrow [\underline{\mathrm{Hom}}(A, B) \otimes C](U) \xrightarrow{\exists \nu(U)} \underline{\mathrm{Hom}}(A, B \otimes C)(U) \\ &= \mathrm{Hom}_{Sh(U)}(A|_U, (B \otimes C)|_U) \end{aligned}$$

by the universal property of tensor products. So we get a sheaf morphism

$$\nu : \underline{\mathrm{Hom}}(A, B) \otimes C \rightarrow \underline{\mathrm{Hom}}(A, B \otimes C).$$

We extend it to complexes of sheaves $A, B, C \in D^b(Z)$ without additional signs since we work with conventions in which $A \otimes \underline{\mathrm{Hom}}^\bullet(A, B) \rightarrow B$ is a chain map. Assuming B to be a complex of injective sheaves we get the required

$$\begin{aligned} \nu : R\underline{\mathrm{Hom}}(A, B) \otimes C &= \underline{\mathrm{Hom}}(A, B) \otimes C \xrightarrow{\nu} \\ &\rightarrow \underline{\mathrm{Hom}}(A, B \otimes C) \rightarrow R\underline{\mathrm{Hom}}(A, B \otimes C). \end{aligned}$$

□

2.2 Corollary. *For $A, B, C, D \in D^b(Z)$ there is an iterated morphism*

$$\begin{aligned} R\underline{\mathrm{Hom}}(A, C) \otimes R\underline{\mathrm{Hom}}(B, D) &\xrightarrow{\nu} R\underline{\mathrm{Hom}}(A, C \otimes R\underline{\mathrm{Hom}}(B, D)) \\ &\xrightarrow{R\underline{\mathrm{Hom}}(A, \nu')} R\underline{\mathrm{Hom}}(A, R\underline{\mathrm{Hom}}(B, C \otimes D)) \xrightarrow{(A.2)} R\underline{\mathrm{Hom}}(A \otimes B, C \otimes D), \end{aligned}$$

where

$$\begin{aligned} \nu' : C \otimes R\mathbf{Hom}(B, D) &\xrightarrow[\sim]{\sigma} R\mathbf{Hom}(B, D) \otimes C \\ &\xrightarrow{\nu} R\mathbf{Hom}(B, D \otimes C) \xrightarrow[\sim]{R\mathbf{Hom}(B, \sigma)} R\mathbf{Hom}(B, C \otimes D), \end{aligned}$$

and σ is the symmetry in the category of complexes – the signed permutation.

2.3 Corollary. For $Z = X \times Y$, $A, C \in D^b(X)$, $B, D \in D^b(Y)$ there is a morphism

$$\begin{aligned} R\mathbf{Hom}(A, C) \boxtimes R\mathbf{Hom}(B, D) &= p_X^* R\mathbf{Hom}(A, C) \otimes p_Y^* R\mathbf{Hom}(B, D) \\ &\xrightarrow{(A.5)} R\mathbf{Hom}(p_X^* A, p_X^* C) \otimes R\mathbf{Hom}(p_Y^* B, p_Y^* D) \\ &\rightarrow R\mathbf{Hom}(p_X^* A \otimes p_Y^* B, p_X^* C \otimes p_Y^* D) = R\mathbf{Hom}(A \boxtimes B, C \boxtimes D), \end{aligned}$$

where the morphism from the previous corollary is used.

2.4 Lemma. Let $f : U \rightarrow X$, $g : V \rightarrow Y$ be continuous maps. For $A \in D^b(X)$, $B \in D^b(Y)$ we have a natural isomorphism

$$(f \times g)^*(A \boxtimes B) \simeq f^* A \boxtimes g^* B.$$

Proof. Using (A.1) we get

$$(f \times g)^*(p_1^* A \otimes p_2^* B) \simeq (f \times g)^* p_1^* A \otimes (f \times g)^* p_2^* B \simeq p_1^* f^* A \otimes p_2^* g^* B.$$

□

2.5 Lemma. Let us consider the following stratified maps of stratified pseudomanifolds.

$$\begin{array}{ccc} S \times Y & \xrightarrow{i \times 1} & U \times Y \\ p_S \downarrow & & \downarrow p_U \\ S & \xrightarrow{i} & U \end{array}$$

For any \mathcal{U} -cohomologically constructible $A \in D_{\mathcal{U}}^{b,c}(U)$ we have

$$(i \times 1)^! p_U^* A \simeq p_S^* i^! A.$$

Proof. Let us prove the Verdier dual of this isomorphism:

$$(i \times 1)^* p_U^! B \simeq p_S^! i^* B$$

for $B = \mathfrak{D}A \in D_{\mathcal{U}}^{b,c}(U)$. Isomorphism (A.7) gives

$$p_U^! B \simeq B \boxtimes \mathfrak{D}_Y \quad \text{and} \quad p_S^! (i^* B) \simeq (i^* B) \boxtimes \mathfrak{D}_Y.$$

Hence,

$$(i \times 1)^* p_U^! B \simeq (i \times 1)^*(B \boxtimes \mathfrak{D}_Y) \simeq (i^* B) \boxtimes \mathfrak{D}_Y \simeq p_S^! i^* B.$$

□

2.6 Proposition. Let (X, \mathcal{X}) be a stratified topological pseudomanifold, and let Y be a locally compact, locally completely paracompact, locally contractible and of finite cohomological dimension over \mathbb{C} . Assume that $A \in D_{\mathcal{X}}^{b,c}(X)$ is \mathcal{X} -cohomologically constructible, $B \in D_{\mathcal{X}}^b(X)$ is \mathcal{X} -cohomologically locally constant, and let $C \in D^b(Y)$. Then the morphism

$$\nu : R\text{Hom}(p_X^*A, p_X^*B) \otimes p_Y^*C \rightarrow R\text{Hom}(p_X^*A, p_X^*B \otimes p_Y^*C).$$

from Lemma 2.1 is an isomorphism.

Proof. Let

$$\mathcal{F}_X : X = X_n \supset X_{n-1} \supset \cdots \supset X_1 \supset X_0 = \emptyset$$

be a closed filtration of X such that the connected components of $S_i = X_i - X_{i-1}$ are strata of X . Let

$$U_i = X - X_{n-i}$$

be the complement open filtration, then $S_{n-k} = U_{k+1} - U_k = X_{n-k} - X_{n-k-1}$. Denote certain inclusions and projections as in the following diagram.

$$\begin{array}{ccccccc} X \times Y & \xleftarrow{J_k} & U_k \times Y & \xhookrightarrow{J} & U_{k+1} \times Y & \xhookrightarrow{I} & S_{n-k} \times Y \\ p_1 \downarrow & & p_1 \downarrow & & p_1 \downarrow & & p_1 \downarrow \\ X & \xleftarrow{j_k} & U_k & \xhookrightarrow{j} & U_{k+1} & \xleftarrow{i} & S_{n-k} \end{array}$$

For a complex $K \in D^b(X \times Y)$ denote $(K)_k = J_k^*K \in D^b(U_k \times Y)$. For $N > n$ we have $(K)_N = K$. We want to prove by induction on k that

$$\nu_k : R\text{Hom}((p_1^*A)_k, (p_1^*B)_k) \otimes (p_2^*C)_k \rightarrow R\text{Hom}((p_1^*A)_k, (p_1^*B)_k \otimes (p_2^*C)_k) \quad (2.1)$$

is an isomorphism. The $k = 1$ (or $n = 1$) case reduces to the simplest situation, where X is a disjoint union of open strata. Since A is cohomologically locally constant with finite dimensional cohomology stalks, it may be replaced *locally* with a complex of constant sheaves with finite dimensional stalks. So $\text{Hom}(p_1^*A, D)$ is isomorphic to a sum of shifted copies of D (with modified differential), hence, it coincides with $R\text{Hom}(p_1^*A, D)$. Clearly, ν_1 is an isomorphism.

For a complex $M \in D^b(X)$ denote $M_k = j_k^*M \in D^b(U_k)$. Then there is an isomorphism for $M \in D^b(X)$

$$(p_1^*M)_k = J_k^*p_1^*M \simeq p_1^*j_k^*M = p_1^*(M_k) = p_1^*M_k.$$

Assuming that ν_k is an isomorphism, let us prove that ν_{k+1} is an isomorphism as well. Apply to the standard triangle

$$I_! I^!(p_1^*B)_{k+1} \rightarrow (p_1^*B)_{k+1} \rightarrow J_*(p_1^*B)_k \rightarrow$$

the both functors in (2.1). It gives two triangles and a morphism between them, written down in diagram in Fig. 1.

$$\begin{array}{ccccccc}
R\underline{\mathrm{Hom}}(p_1^*A_{k+1}, I_! I_1^! p_1^*B_{k+1}) \otimes (p_2^*C)_{k+1} & \longrightarrow & R\underline{\mathrm{Hom}}(p_1^*A_{k+1}, p_1^*B_{k+1}) \otimes (p_2^*C)_{k+1} & \longrightarrow & R\underline{\mathrm{Hom}}(p_1^*A_{k+1}, J_* p_1^*B_k) \otimes (p_2^*C)_{k+1} & \longrightarrow & \\
\nu' \downarrow & & \nu_{k+1} \downarrow & & \nu'' \downarrow & & \\
R\underline{\mathrm{Hom}}(p_1^*A_{k+1}, I_! I_1^! p_1^*B_{k+1} \otimes (p_2^*C)_{k+1}) & \longrightarrow & R\underline{\mathrm{Hom}}(p_1^*A_{k+1}, p_1^*B_{k+1} \otimes (p_2^*C)_{k+1}) & \longrightarrow & R\underline{\mathrm{Hom}}(p_1^*A_{k+1}, J_* p_1^*B_k \otimes (p_2^*C)_{k+1}) & \longrightarrow &
\end{array}$$

Figure 1: A morphism of two distinguished triangles

Let us prove that ν'' is an isomorphism. Indeed, this morphism is a composition of several isomorphisms:

$$\begin{aligned}
& R\mathbf{Hom}(p_1^*A_{k+1}, J_*p_1^*B_k) \otimes (p_2^*C)_{k+1} \xrightarrow[\sim]{(A.3)} \\
& J_*R\mathbf{Hom}(p_1^*A_k, p_1^*B_k) \otimes (p_2^*C)_{k+1} \xrightarrow[\sim]{(A.5)} \\
& J_*p_1^*R\mathbf{Hom}(A_k, B_k) \otimes (p_2^*C)_{k+1} \xrightarrow[\sim]{} \\
& J_*p_1^*R\mathbf{Hom}(A, B)_k \otimes (p_2^*C)_{k+1} \xrightarrow[\sim]{}
\end{aligned}$$

(since $R\mathbf{Hom}(A, B)$ is \mathcal{X} -cohomologically locally constant by [3] Theorem 8.6, we can apply Lemma 10.22 loc. cit.)

$$\begin{aligned}
& J_*[p_1^*R\mathbf{Hom}(A, B)_k \otimes (p_2^*C)_k] \xrightarrow[\sim]{J_*\nu_k} \\
& J_*[R\mathbf{Hom}(p_1^*A_k, p_1^*B_k \otimes (p_2^*C)_k)] \xrightarrow[\sim]{(A.3)} \\
& R\mathbf{Hom}(p_1^*A_{k+1}, J_*[p_1^*B_k \otimes (p_2^*C)_k]) \xrightarrow[\sim]{}
\end{aligned}$$

(again by Lemma 10.22 [3])

$$R\mathbf{Hom}(p_1^*A_{k+1}, J_*p_1^*B_k \otimes (p_2^*C)_{k+1}).$$

Let us prove that ν' is an isomorphism. Indeed, this morphism is a composition of several isomorphisms:

$$\begin{aligned}
& R\mathbf{Hom}(p_1^*A_{k+1}, I_!I^!p_1^*B_{k+1}) \otimes (p_2^*C)_{k+1} \xrightarrow[\sim]{(A.3)} \\
& I_!R\mathbf{Hom}(I^*p_1^*A_{k+1}, I^!p_1^*B_{k+1}) \otimes (p_2^*C)_{k+1} \xrightarrow[\sim]{(A.4)} \\
& I_![R\mathbf{Hom}(I^*p_1^*A_{k+1}, I^!p_1^*B_{k+1}) \otimes I^*(p_2^*C)_{k+1}] \xrightarrow[\sim]{}
\end{aligned}$$

(by Lemma 2.5 with $S = S_{n-k}$, $U = U_{k+1}$, $A = B_{k+1}$)

$$I_![R\mathbf{Hom}(p_1^*i^*A_{k+1}, p_1^*i^!(B_{k+1})) \otimes p_2^*C] \xrightarrow[\sim]{I_!\nu}$$

(applying $n = 1$ case to S_{n-k} with the trivial filtration in place of X)

$$\begin{aligned}
& I_![R\mathbf{Hom}(p_1^*i^*A_{k+1}, p_1^*i^!(B_{k+1}) \otimes p_2^*C)] \xrightarrow[\sim]{\text{Lemma 2.5}} \\
& I_![R\mathbf{Hom}(I^*p_1^*A_{k+1}, I^!p_1^*B_{k+1} \otimes I^*(p_2^*C)_{k+1})] \xrightarrow[\sim]{(A.3)} \\
& R\mathbf{Hom}(p_1^*A_{k+1}, I_![I^!p_1^*B_{k+1} \otimes I^*(p_2^*C)_{k+1}]) \xrightarrow[\sim]{(A.4)} \\
& R\mathbf{Hom}(p_1^*A_{k+1}, I_!I^!p_1^*B_{k+1} \otimes (p_2^*C)_{k+1}).
\end{aligned}$$

Since ν' and ν'' are isomorphisms, so is ν_{k+1} . □

The above proposition determines when the morphisms ν used in Corollary 2.3 are isomorphisms. So we get

2.7 Corollary. *Let $A \in D_{\mathcal{X}}^{b,c}(X)$ be \mathcal{X} -cohomologically constructible, let $B \in D_{\mathcal{Y}}^{b,c}(Y)$ be \mathcal{Y} -cohomologically constructible, let $C \in D_{\mathcal{X}}^b(X)$ be \mathcal{X} -cohomologically locally constant, and let $D \in D_{\mathcal{Y}}^b(Y)$ be \mathcal{Y} -cohomologically locally constant. Then the morphism*

$$R\mathbf{Hom}(A, C) \boxtimes R\mathbf{Hom}(B, D) \rightarrow R\mathbf{Hom}(A \boxtimes B, C \boxtimes D)$$

is an isomorphism.

2.8 Corollary. *Let $A \in D_{\mathcal{X}}^{b,c}(X)$ be \mathcal{X} -cohomologically constructible, and let $B \in D_{\mathcal{Y}}^{b,c}(Y)$ be \mathcal{Y} -cohomologically constructible. Then*

$$\mathfrak{D}A \boxtimes \mathfrak{D}B \simeq \mathfrak{D}(A \boxtimes B).$$

Proof. Indeed,

$$\begin{aligned} \mathfrak{D}A \boxtimes \mathfrak{D}B &= R\mathbf{Hom}(A, \mathfrak{D}_X) \boxtimes R\mathbf{Hom}(B, \mathfrak{D}_Y) \\ &\xrightarrow[\sim]{\text{Corollary 2.7}} R\mathbf{Hom}(A \boxtimes B, \mathfrak{D}_X \boxtimes \mathfrak{D}_Y) \\ &\xrightarrow[\sim]{(\text{A.8})} R\mathbf{Hom}(A \boxtimes B, \mathfrak{D}_{X \times Y}) \\ &= \mathfrak{D}(A \boxtimes B). \end{aligned}$$

□

In addition to Lemma 2.4 we have

2.9 Proposition. (i) *Let $A \in D_{\mathcal{X}}^{b,c}(X)$, $B \in D_{\mathcal{Y}}^{b,c}(Y)$, and let $f : U \rightarrow X$, $g : V \rightarrow Y$ be stratified maps. Then*

$$(f \times g)^!(A \boxtimes B) \simeq f^!A \boxtimes g^!B.$$

(ii) (a) *Let $A \in D^b(X)$, $B \in D^b(Y)$, and let $f : X \rightarrow U$, $g : Y \rightarrow V$ be continuous maps. Then*

$$(f \times g)_!(A \boxtimes B) \simeq f_!A \boxtimes g_!B.$$

(b) *Furthermore, if $A \in D_{\mathcal{X}}^{b,c}(X)$, $B \in D_{\mathcal{Y}}^{b,c}(Y)$, and the stratified maps f, g are proper, or complex algebraic, or every fibre of f, g is compactifiable, then*

$$(f \times g)_*(A \boxtimes B) \simeq f_*A \boxtimes g_*B.$$

Proof. (i) Deduce this isomorphism applied to the objects $\mathfrak{D}A, \mathfrak{D}B$ via Lemma 2.4:

$$\begin{aligned} (f \times g)^!(\mathfrak{D}A \boxtimes \mathfrak{D}B) &\xrightarrow[\sim]{\text{Corollary 2.8}} (f \times g)^!\mathfrak{D}(A \boxtimes B) \\ &\xrightarrow[\sim]{(\text{A.9})} \mathfrak{D}(f \times g)^*(A \boxtimes B) \\ &\xrightarrow[\sim]{\text{Lemma 2.4}} \mathfrak{D}(f^*A \boxtimes g^*B) \\ &\xrightarrow[\sim]{\text{Corollary 2.8}} \mathfrak{D}f^*A \boxtimes \mathfrak{D}g^*B \\ &\xrightarrow[\sim]{(\text{A.9})} f^!\mathfrak{D}A \boxtimes g^!\mathfrak{D}B. \end{aligned}$$

(ii)(a) It suffices to consider the case $g = \text{id}$. Combined with the similar case $f = \text{id}$, it implies the general case. Based on the diagram

$$\begin{array}{ccccc} X & \xleftarrow{p_X} & X \times Y & \xrightarrow{p_2} & Y \\ \downarrow f & & \downarrow f \times 1 & & \parallel \\ U & \xleftarrow{p_U} & U \times Y & \xrightarrow{p_2} & Y \end{array}$$

the required isomorphism is composed of the following isomorphisms

$$\begin{aligned} (f \times 1)_!(A \boxtimes B) &= (f \times 1)_!(p_X^* A \otimes p_2^* B) \\ &\simeq (f \times 1)_!(p_X^* A \otimes (f \times 1)^* p_2^* B) \\ &\xrightarrow[\sim]{(A.4)} (f \times 1)_! p_X^* A \otimes p_2^* B \\ &\xrightarrow[\sim]{\text{base change}} p_U^* f_! A \otimes p_2^* B \\ &= f_! A \boxtimes B. \end{aligned}$$

(ii)(b) Is deduced from (ii)(a) using (A.10) similarly to (i). □

2.10 Proposition. *The functor*

$$\boxtimes : {}^p D_{\mathcal{X}}^{b,c}(X) \times {}^q D_{\mathcal{Y}}^{b,c}(Y) \rightarrow {}^{p+q} D_{\mathcal{X} \times \mathcal{Y}}^{b,c}(X \times Y)$$

is *t-exact*.

Proof. For $K \in D^b(X)$, $L \in D^b(Y)$ we have the Künneth formula

$$\begin{aligned} H^n(K \boxtimes L) &= H^n(p_1^* K \otimes p_2^* L) \simeq \bigoplus_{k+l=n} H^k p_1^* K \otimes H^l p_2^* L \\ &\simeq \bigoplus_{k+l=n} p_1^* H^k K \otimes p_2^* H^l L = \bigoplus_{k+l=n} H^k K \boxtimes H^l L. \end{aligned}$$

Let $A \in D_{\mathcal{X}}^{\leq p}(X)$, $B \in D_{\mathcal{Y}}^{\leq q}(Y)$, and let $S \in \mathcal{X}$, $T \in \mathcal{Y}$ be strata. If $n > p(S) + q(T)$, then

$$H^n(i_S \times i_T)^*(A \boxtimes B) = H^n(i_S^* A \boxtimes i_T^* B) \simeq \bigoplus_{k+l=n} H^k(i_S^* A) \boxtimes H^l(i_T^* B) = 0$$

by Lemma 2.4 and by the Künneth formula. Hence, $A \boxtimes B \in D_{\mathcal{X} \times \mathcal{Y}}^{\leq p+q}(X \times Y)$.

Suppose now that $A \in D_{\mathcal{X}}^{\geq p}(X)$, $B \in D_{\mathcal{Y}}^{\geq q}(Y)$. If $n < p(S) + q(T)$, then

$$H^n(i_S \times i_T)^!(A \boxtimes B) = H^n(i_S^! A \boxtimes i_T^! B) \simeq \bigoplus_{k+l=n} H^k(i_S^! A) \boxtimes H^l(i_T^! B) = 0$$

by Proposition 2.9 and the Künneth formula. Hence, $A \boxtimes B \in D_{\mathcal{X} \times \mathcal{Y}}^{\geq p+q}(X \times Y)$. □

2.11 Corollary. *The restriction of \boxtimes to perverse sheaves gives a \mathbb{C} -bilinear functor*

$$\boxtimes : \text{Perv}(X, \mathcal{X}, p) \times \text{Perv}(Y, \mathcal{Y}, q) \rightarrow \text{Perv}(X \times Y, \mathcal{X} \times \mathcal{Y}, p + q)$$

exact in each variable.

Proof. For a fixed $B \in \text{Perv}(Y)$ the functor $T : {}^p D_{\mathcal{X}}^{b,c}(X) \rightarrow {}^{p+q} D_{\mathcal{X} \times \mathcal{Y}}^{b,c}(X \times Y)$, $A \mapsto A \boxtimes B$ is *t-exact*. Hence, the functor

$${}^p T = {}^{p+q} H^0 \circ T \circ \epsilon : \text{Perv}(X) \rightarrow \text{Perv}(Y), \quad A \mapsto A \boxtimes B,$$

is exact by [1, Proposition 1.3.17(i)]. □

3. Simple perverse sheaves

3.1. The case of trivial stratification

In the case of a trivial stratification $\mathcal{X} = \{X\}$ of a connected manifold X the perversity p is an integer $p(X)$. We have

$$\begin{aligned} \text{Perv}(X, \mathcal{X}, p) &= \{K \in D_{\mathcal{X}}^{b,c}(X) \mid H^n K = 0 \text{ unless } n = p(X)\} \\ &= \mathcal{LCS}h(X)[-p(X)] \end{aligned}$$

and the category of locally constant sheaves of finite rank $\mathcal{LCS}h(X)$ is equivalent to $\pi_1(X)$ -mod – the category of $\pi_1(X)$ -modules finite dimensional over \mathbb{C} .

3.2. Intersection cohomology sheaves

By [1, Proposition 1.4.26] any simple perverse sheaf on (X, \mathcal{X}, p) comes from a simple $\pi_1(S)$ -module for one of the strata S via the functor

$$\begin{aligned} \pi_1(S)\text{-mod} &\xrightarrow{\sim} \mathcal{LCS}h(S) \xrightarrow{\sim} \mathcal{LCS}h(S)[-p(S)] \\ &= \text{Perv}(S, \{S\}, p(S)) \xrightarrow{j_{S!}*} \text{Perv}(\overline{S}) \xrightarrow{i_{S*}} \text{Perv}(X), \end{aligned}$$

where \overline{S} is the closure of S and $j_S : S \hookrightarrow \overline{S}$, $i_S : \overline{S} \hookrightarrow X$ are the inclusions. The prolongation functor $j_{S!}*$ is defined in [1, Definition 1.4.22] (see Definition A.1). Here the functor i_{S*} is the restriction of a t -exact functor $i_{S*} : D_{\overline{S}}^{b,c}(\overline{S}) \rightarrow D_{\mathcal{X}}^{b,c}(X)$, $\overline{S} = \mathcal{X} \cap \overline{S}$ [1, Proposition 1.4.16].

Now let us discuss the behaviour of so obtained perverse sheaves with respect to \boxtimes .

3.3 Proposition. *Let $S \in \mathcal{X}$, $T \in \mathcal{Y}$ be strata of (X, \mathcal{X}, p) and (Y, \mathcal{Y}, q) . Then there are functorial isomorphisms*

$$\begin{array}{ccccc} \pi_1(S)\text{-mod} \times \pi_1(T)\text{-mod} & \xrightarrow{\otimes_{\mathbb{C}}} & \pi_1(S \times T)\text{-mod} \\ \downarrow \wr & \simeq & \downarrow \wr \\ \mathcal{LCS}h(S) \times \mathcal{LCS}h(T) & \xrightarrow{\boxtimes} & \mathcal{LCS}h(S \times T) \\ \downarrow \wr & \simeq & \downarrow \wr \\ \mathcal{LCS}h(S)[-p(S)] \times \mathcal{LCS}h(T)[-q(T)] & \xrightarrow{\boxtimes} & \mathcal{LCS}h(S \times T)[-p(S) - q(T)] \\ \parallel & = & \parallel \\ \text{Perv}(S) \times \text{Perv}(T) & \xrightarrow{\boxtimes} & \text{Perv}(S \times T) \\ \downarrow j_{S!}* \times j_{T!}* & \simeq & \downarrow (j_S \times j_T)!* \\ \text{Perv}(\overline{S}) \times \text{Perv}(\overline{T}) & \xrightarrow{\boxtimes} & \text{Perv}(\overline{S} \times \overline{T}) \\ \downarrow i_{S*} \times i_{T*} & \simeq & \downarrow (i_S \times i_T)* \\ \text{Perv}(X) \times \text{Perv}(Y) & \xrightarrow{\boxtimes} & \text{Perv}(X \times Y) \end{array}$$

Proof. The essential part is to construct a functorial isomorphism

$$(j_S \times j_T)_{!*}(A \boxtimes B) \simeq j_{S!*}A \boxtimes j_{T!*}B.$$

Due to the isomorphism

$$(j_S \times j_T)_{!*} \simeq (j_S \times 1)_{!*}(1 \times j_T)_{!*},$$

which follows from [1, (2.1.7.1)], see (A.11), we only have to prove the following lemma. \square

3.4 Lemma. *Let U be an open stratified subspace of a stratified pseudomanifold $Z = (Z, \mathcal{Z}, p)$. Denote $j : U \subset Z$ the inclusion. Let $A \in \text{Perv}(U, p)$ and $B \in \text{Perv}(W, q)$. Then in $\text{Perv}(Z \times W, p \dot{+} q)$ we have a functorial isomorphism*

$$(j \times 1)_{!*}(A \boxtimes B) \simeq j_{!*}A \boxtimes B.$$

Proof. The left hand side is determined uniquely as a prolongation C of $A \boxtimes B$ to $D_{Z \times W}^{b,c}(Z \times W)$, that is a complex equipped with an isomorphism $(j \times 1)^*C \simeq A \boxtimes B$, such that for any stratum $s : S \hookrightarrow Z$, that is not contained in U , and for any stratum $t : T \hookrightarrow W$ we have $H^m((s \times t)^*C) = 0$ for $m \geq p(S) + q(T)$ and $H^m((s \times t)^!C) = 0$ for $m \leq p(S) + q(T)$ [1, Proposition 2.1.9], see Definition A.1.

Let us verify these conditions for the right hand side. Indeed, we have

$$(j \times 1)^*(j_{!*}A \times B) \simeq j^*j_{!*}A \boxtimes B \simeq A \boxtimes B, \quad (3.1)$$

$$\begin{aligned} H^m(s \times t)^*(j_{!*}A \times B) &\simeq H^m(s^*j_{!*}A \boxtimes t^*B) \\ &\simeq \bigoplus_{k+l=m} H^k(s^*j_{!*}A) \boxtimes H^l(t^*B) \\ &\simeq \bigoplus_{\substack{k+l=m \\ k < p(S), l \leq q(T)}} H^k(s^*j_{!*}A) \boxtimes H^l(t^*B). \end{aligned}$$

This vanishes if $m \geq p(S) + q(T)$. Similarly,

$$\begin{aligned} H^m(s \times t)^!(j_{!*}A \times B) &\simeq H^m(s^!j_{!*}A \boxtimes t^!B) \\ &\simeq \bigoplus_{k+l=m} H^k(s^!j_{!*}A) \boxtimes H^l(t^!B) \\ &\simeq \bigoplus_{\substack{k+l=m \\ k > p(S), l \geq q(T)}} H^k(s^!j_{!*}A) \boxtimes H^l(t^!B). \end{aligned}$$

This vanishes if $m \leq p(S) + q(T)$.

Therefore, there exists a unique isomorphism

$$\alpha_{A,B} : (j \times 1)_{!*}(A \boxtimes B) \xrightarrow{\sim} j_{!*}A \boxtimes B,$$

such that the diagram

$$\begin{array}{ccc} (j \times 1)^*(j \times 1)_{!*}(A \boxtimes B) & \xrightarrow[\sim]{(j \times 1)^*\alpha_{A,B}} & (j \times 1)^*(j_{!*}A \boxtimes B), \\ \downarrow \wr & & \downarrow \wr (3.1) \\ A \boxtimes B & \xlongequal{\hspace{1.5cm}} & A \boxtimes B \end{array}$$

is commutative. Uniqueness of $\alpha_{A,B}$ implies its functoriality in A, B . \square

4. Right derived Hom^\bullet and the external tensor product

The right derived Hom^\bullet can be defined as the functor

$$R\mathrm{Hom}^\bullet : D(X)^{\mathrm{op}} \times D(X) \rightarrow D(\mathbb{C}\text{-Vect})$$

equipped with an isomorphism

$$\begin{array}{ccc} K(\mathcal{S}h(X))^{\mathrm{op}} \times K(\mathcal{S}h(X)) & \xrightarrow{\mathrm{Hom}^\bullet} & K(\mathbb{C}\text{-Vect}) \\ \downarrow & & \downarrow \\ D(X)^{\mathrm{op}} \times D(X) & \xrightarrow{R\mathrm{Hom}^\bullet} & D(\mathbb{C}\text{-Vect}) \end{array}$$

Its value on $A, B \in D(X)$ can be computed as $\mathrm{Hom}^\bullet(A, I)$ for a quasiisomorphism $B \rightarrow I$ with a complex I of injective sheaves. The object $R\mathrm{Hom}(A, B) \in D(X)$ also can be computed as $\underline{\mathrm{Hom}}(A, I)$, which is a complex of flabby sheaves. To compute the value of $R\Gamma = Rp_* : D(X) \rightarrow D(\mathbb{C}\text{-Vect})$ for $p : X \rightarrow pt$ on $R\mathrm{Hom}(A, B)$ we can use the flabby resolution $\underline{\mathrm{Hom}}(A, I)$ of $R\mathrm{Hom}(A, B)$. Hence,

$$Rp_* R\mathrm{Hom}(A, B) \simeq p_* \underline{\mathrm{Hom}}(A, I) = \mathrm{Hom}^\bullet(A, I) \simeq R\mathrm{Hom}^\bullet(A, B),$$

or $R\Gamma \circ R\mathrm{Hom} \simeq R\mathrm{Hom}^\bullet$.

Let us apply Proposition 2.9(ii)(b) to maps $p_X : X \rightarrow pt$, $p_Y : Y \rightarrow pt$. Recall that the stratified pseudomanifolds X, Y are assumed in this paper to be compactifiable (for instance, compact or complex algebraic).

4.1 Lemma. For arbitrary $E \in D_X^{b,c}(X)$ and $F \in D_Y^{b,c}(Y)$ we have

$$R\Gamma(E \boxtimes F) \simeq R\Gamma E \otimes_{\mathbb{C}} R\Gamma F.$$

Proof. $(p_X \times p_Y)_*(E \boxtimes F) \simeq p_{X*}E \boxtimes p_{Y*}F = p_{X*}E \otimes_{\mathbb{C}} p_{Y*}F$. \square

4.2 Corollary. For $A, B \in D_X^{b,c}(X)$ and $C, D \in D_Y^{b,c}(Y)$ there is a functorial isomorphism

$$R\mathrm{Hom}^\bullet(A, B) \otimes_{\mathbb{C}} R\mathrm{Hom}^\bullet(C, D) \xrightarrow{\sim} R\mathrm{Hom}^\bullet(A \boxtimes C, B \boxtimes D).$$

Proof. Follows from Lemma 4.1 with $E = \underline{\mathrm{Hom}}(A, B)$ and $F = \underline{\mathrm{Hom}}(C, D)$ and Corollary 2.7. \square

Applying the Künneth formula we get

4.3 Corollary. For \mathcal{X} -cohomologically constructible A, B and \mathcal{Y} -cohomologically constructible C, D we have an isomorphism

$$\oplus_{i+j=k} \mathrm{Ext}_{D(X)}^i(A, B) \otimes_{\mathbb{C}} \mathrm{Ext}_{D(Y)}^j(C, D) \xrightarrow{\sim} \mathrm{Ext}_{D(X \times Y)}^k(A \boxtimes C, B \boxtimes D).$$

4.4 Corollary. If $A, B \in \mathrm{Perv}(X)$, $C, D \in \mathrm{Perv}(Y)$, then

$$\mathrm{Hom}_{\mathrm{Perv}(X)}(A, B) \otimes_{\mathbb{C}} \mathrm{Hom}_{\mathrm{Perv}(Y)}(C, D) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Perv}(X \times Y)}(A \boxtimes C, B \boxtimes D). \quad (4.1)$$

Indeed, for perverse sheaves $\mathrm{Ext}_{D(X)}^i(A, B) = 0$ for $i < 0$ by [1, Corollaire 2.1.4].

5. Deligne's external tensor product of perverse sheaves

It follows from [1], Amplification 1.4.17.1 and Proposition 1.4.18, by induction on strata that any object of the abelian category $\text{Perv}(X)$ has finite length. Constructibility of perverse sheaves implies that the \mathbb{C} -vector spaces $\text{Hom}_{\text{Perv}(X)}(A, B)$ are finite dimensional. Therefore, $\text{Perv}(X)$ is an inductive limit of its full subcategories $\langle M \rangle$, equivalent to $A\text{-mod}$ for some finite dimensional associative unital algebra A [5, Corollaire 2.17]. The subcategory $\langle M \rangle$ is formed by subquotients of M^n , $n \in \mathbb{Z}_{>0}$. By results of Deligne [5] there exists an (abstract) external tensor product of the categories $\text{Perv}(X)$ and $\text{Perv}(Y)$ – the universal functor

$$\boxtimes^D : \text{Perv}(X) \times \text{Perv}(Y) \rightarrow \text{Perv}(X) \boxtimes^D \text{Perv}(Y),$$

whose target is some \mathbb{C} -linear abelian category. Universality implies, in particular, that there exists an exact \mathbb{C} -linear functor F and an isomorphism

$$\begin{array}{ccc} \text{Perv}(X) \times \text{Perv}(Y) & \xrightarrow{\boxtimes^D} & \text{Perv}(X) \boxtimes^D \text{Perv}(Y) \\ & \searrow \boxtimes & \downarrow F \\ & & \text{Perv}(X \times Y) \end{array} \quad (5.1)$$

Our goal is to prove that F is an equivalence. Once this is done, we can choose the Deligne external tensor product \boxtimes^D to be the geometric external tensor product \boxtimes and $\text{Perv}(X) \boxtimes^D \text{Perv}(Y)$ to be $\text{Perv}(X \times Y)$. Thus, we can use the same notation \boxtimes in both abstract and geometric senses.

5.1 Theorem. *The functor*

$$F : \text{Perv}(X) \boxtimes^D \text{Perv}(Y) \rightarrow \text{Perv}(X \times Y)$$

determined by diagram (5.1) is an equivalence. Therefore, we can choose as tensor product $\text{Perv}(X) \boxtimes^D \text{Perv}(Y) = \text{Perv}(X \times Y)$ and $\boxtimes^D = \boxtimes$.

Proof. The results of Section 3 describe $\text{SimpPerv}(X)$ – the list of isomorphism classes of simple objects of $\text{Perv}(X)$. For $\text{Perv}(X) \boxtimes^D \text{Perv}(Y)$ this list is the tensor product $\text{SimpPerv}(X) \boxtimes^D \text{SimpPerv}(Y)$ of the lists for X and Y [5, Lemme 5.9] due to algebraic closedness of \mathbb{C} . On the other hand, Proposition 3.3 implies that

$$\text{SimpPerv}(X \times Y) = \text{SimpPerv}(X) \boxtimes \text{SimpPerv}(Y).$$

Therefore, the functor F maps bijectively the list of isomorphism classes of simple objects of $\text{Perv}(X) \boxtimes^D \text{Perv}(Y)$ to the list of isomorphism classes of simple perverse sheaves of $\text{Perv}(X \times Y)$.

5.2 Lemma ([2] Lemma 3.2.4). *The natural mapping of Yoneda's Ext to the Ext in the derived category*

$$\theta : {}^Y \text{Ext}_{\text{Perv}(X)}^i(A, B) \rightarrow \text{Ext}_{D(X)}^i(A, B) \stackrel{\text{def}}{=} \text{Hom}_{D(X)}(A, B[i])$$

is bijective for $k = 0, 1$ and injective for $k = 2$ for all $A, B \in \text{Perv}(X)$.

5.3 Lemma. *Let \mathcal{A}, \mathcal{B} be \mathbb{C} -linear abelian categories with length (that is objects have finite length and Hom spaces are finite dimensional). Then*

$$\bigoplus_{i+j=k} {}^Y\text{Ext}_{\mathcal{A}}^i(K, L) \otimes_{\mathbb{C}} {}^Y\text{Ext}_{\mathcal{B}}^j(M, N) \xrightarrow{\sim} {}^Y\text{Ext}_{\mathcal{A} \boxtimes^D \mathcal{B}}^k(K \boxtimes^D M, L \boxtimes^D N).$$

Proof. From the definition of Yoneda's ${}^Y\text{Ext}$ [10] it is clear that

$${}^Y\text{Ext}_{\text{Perv}(X)}^i(A, B) = \varinjlim_{\text{Perv}(X) \supset \langle P \rangle \ni A, B} {}^Y\text{Ext}_{\langle P \rangle}^i(A, B), \quad (5.2)$$

where P runs over such objects of $\text{Perv}(X)$ that the subcategory $\langle P \rangle$ contains A and B . Since the category $\langle P \rangle$ has enough injectives and projectives, we can identify ${}^Y\text{Ext}_{\langle P \rangle}^i(C, D)$ with the right derived functor $\text{Ext}_{\langle P \rangle}^i(C, D)$ of $\text{Hom}_{\langle P \rangle}$.

It suffices to prove the statement for subcategories of \mathcal{A}, \mathcal{B} of the form $\langle P \rangle$. So we have to show that the external product map

$$\bigoplus_{i+j=k} \text{Ext}_A^i(K, L) \otimes_{\mathbb{C}} \text{Ext}_B^j(M, N) \longrightarrow \text{Ext}_{A \otimes_{\mathbb{C}} B}^k(K \otimes_{\mathbb{C}} M, L \otimes_{\mathbb{C}} N)$$

is bijective for finite dimensional \mathbb{C} -algebras A, B , finite dimensional A -modules K, L and finite dimensional B -modules M, N . This is precisely one of the assertions of Theorem XI.3.1 of Cartan and Eilenberg [4]. \square

Combining the above lemmas and Corollary 4.3 we get a commutative diagram

$$\begin{array}{ccc} {}^Y\text{Ext}_{\text{Perv}(X) \boxtimes^D \text{Perv}(Y)}^k(A \boxtimes^D C, B \boxtimes^D D) & & \\ \downarrow \text{Lemma 5.3} \wr & & \\ \bigoplus_{i+j=k} {}^Y\text{Ext}_{\text{Perv}(X)}^i(A, B) \otimes_{\mathbb{C}} {}^Y\text{Ext}_{\text{Perv}(Y)}^j(C, D) & \xrightarrow{\theta} & \bigoplus_{i+j=k} \text{Ext}_{D(X)}^i(A, B) \otimes_{\mathbb{C}} \text{Ext}_{D(Y)}^j(C, D) \\ \downarrow \text{V} & & \downarrow \wr \text{Corollary 4.3} \\ {}^Y\text{Ext}_{\text{Perv}(X \times Y)}^k(A \boxtimes C, B \boxtimes D) & \xrightarrow{\theta} & \text{Ext}_{D(X \times Y)}^k(A \boxtimes C, B \boxtimes D) \end{array}$$

for all $A, B \in \text{Perv}(X)$, $C, D \in \text{Perv}(Y)$ and all $k \in \mathbb{Z}_{\geq 0}$. Here the map V takes

$$[0 \rightarrow B \rightarrow M_1 \rightarrow \dots \rightarrow M_i \rightarrow A \rightarrow 0] \otimes [0 \rightarrow D \rightarrow N_1 \rightarrow \dots \rightarrow N_j \rightarrow C \rightarrow 0]$$

to

$$\begin{aligned} [0 \rightarrow B \boxtimes D \rightarrow M_1 \boxtimes D \rightarrow \dots \rightarrow M_i \boxtimes D \rightarrow \\ \rightarrow A \boxtimes N_1 \rightarrow \dots \rightarrow A \boxtimes N_j \rightarrow A \boxtimes C \rightarrow 0] \end{aligned}$$

where the middle map is the composition of $M_i \boxtimes D \rightarrow A \boxtimes D \rightarrow A \boxtimes N_1$. Commutativity of the square follows from our sign convention. It is verified similarly to Yoneda's computation [11] of the V multiplication of Cartan and Eilenberg [4, Section XI.1].

By Lemma 5.2 the horizontal arrows are bijective for $k = 0, 1$ and injective for $k = 2$. Hence, the same holds for the mapping V . In particular, the induced by F mapping of Yoneda's ${}^Y\text{Ext}^k$ between simple objects of $\text{Perv}(X) \boxtimes^D \text{Perv}(Y)$ to that of $\text{Perv}(X \times Y)$ is bijective for $k = 0, 1$ and injective for $k = 2$. It remains to apply the following

5.4 Lemma. *Let $F : \mathcal{A} \rightarrow \mathcal{B}$ be an exact functor between essentially small categories with length. Assume that F induces a bijection on the list of isomorphism classes of simple objects. Assume also that the maps induced by F*

$${}^Y\mathrm{Ext}_{\mathcal{A}}^k(T, S) \rightarrow {}^Y\mathrm{Ext}_{\mathcal{B}}^k(FT, FS) \quad (5.3)$$

are bijective for $k = 0, 1$ and injective for $k = 2$ for all simple objects T, S of \mathcal{A} . Then F is an equivalence.

Proof. First we prove by induction on the length of T that (5.3) is an isomorphism for $k = 0, 1$ for a simple S and an arbitrary T . Indeed, write the long exact sequences up to $k = 2$ for ${}^Y\mathrm{Ext}_{\mathcal{A}}^{\bullet}(-, S)$ and ${}^Y\mathrm{Ext}_{\mathcal{B}}^{\bullet}(F-, FS)$ associated with $0 \rightarrow T' \rightarrow T \rightarrow T'' \rightarrow 0$, where T'' is simple, and use the 5-Lemma. Second, we prove by induction on the length of S that (5.3) is an isomorphism for $k = 0$ for all objects S, T of \mathcal{A} . Indeed, write the long exact sequences up to $k = 1$ for ${}^Y\mathrm{Ext}_{\mathcal{A}}^{\bullet}(T, -)$ and ${}^Y\mathrm{Ext}_{\mathcal{B}}^{\bullet}(FT, F-)$ associated with $0 \rightarrow S' \rightarrow S \rightarrow S'' \rightarrow 0$, where S' is simple, and use the 5-Lemma. Hence, F is full and faithful.

Now we prove by induction on length that F induces surjection on the set of isomorphism classes of objects of length $\leq n$ for \mathcal{A} and \mathcal{B} . For $n = 1$ it is the hypothesis of the lemma. For $X \in \mathrm{Ob} \mathcal{B}$ of length $n > 1$ we can assume the existence of

$$0 \rightarrow FS \rightarrow X \rightarrow FT \rightarrow 0 \quad (5.4)$$

for some simple $S \in \mathrm{Ob} \mathcal{A}$ and some $T \in \mathrm{Ob} \mathcal{A}$ of length less than n . Since the map

$${}^Y\mathrm{Ext}_{\mathcal{A}}^1(T, S) \rightarrow {}^Y\mathrm{Ext}_{\mathcal{B}}^1(FT, FS)$$

is bijective, there exists a short exact sequence $0 \rightarrow S \rightarrow Y \rightarrow T \rightarrow 0$ in \mathcal{A} such that $0 \rightarrow FS \rightarrow FY \rightarrow FT \rightarrow 0$ is congruent with (5.4). In particular, $X \simeq FY$.

Therefore, F is full, faithful and essentially surjective on objects. \square

Applying this lemma to $\mathcal{A} = \mathrm{Perv}(X) \boxtimes^D \mathrm{Perv}(Y)$ and $\mathcal{B} = \mathrm{Perv}(X \times Y)$ we prove the theorem. \square

A. Some formulas

Here we summarise some formulas taken mostly from Borel [3] §10. All spaces are locally compact, locally completely paracompact, locally contractible and of finite cohomological dimension over \mathbb{C} .

For a continuous map $f : X \rightarrow Y$ and $A, B \in D(Y)$ we have

$$f^*(A \otimes B) \simeq f^*A \otimes f^*B, \quad (\text{A.1})$$

by loc. cit. Proposition 10.1.

For $A, B, C \in D(X)$ we have

$$R\underline{\mathrm{Hom}}(A \otimes B, C) \simeq R\underline{\mathrm{Hom}}(A, R\underline{\mathrm{Hom}}(B, C)) \quad (\text{A.2})$$

by loc. cit. Proposition 10.2.

For a continuous map $f : X \rightarrow Y$ and $A \in D(Y)$, $B \in D(X)$ we have

$$f_* R\mathbf{Hom}(f^* A, B) \simeq R\mathbf{Hom}(A, f_* B), \quad (\text{A.3})$$

by loc. cit. Proposition 10.3(1) and

$$f_!(B \otimes f^* A) \simeq f_! B \otimes A, \quad (\text{A.4})$$

by loc. cit. Proposition 10.8(2).

Denoting $p_1 : X \times Y \rightarrow X$ the projection on the first space, we have for $A, B \in D^b(X)$

$$p_1^* R\mathbf{Hom}(A, B) \simeq R\mathbf{Hom}(p_1^* A, p_1^* B) \quad (\text{A.5})$$

by loc. cit. Proposition 10.21.

Denoting $p_2 : X \times Y \rightarrow Y$ the projection on the second space, we have for \mathcal{X} -cohomologically locally constant $A \in D_{\mathcal{X}}^b(X)$ and \mathcal{Y} -cohomologically constructible $B \in D_{\mathcal{Y}}^{b,c}(Y)$

$$p_1^* \mathfrak{D} A \otimes p_2^* B \simeq R\mathbf{Hom}(p_1^* A, p_2^! B) \quad (\text{A.6})$$

by loc. cit. Theorem 10.25.

Here

$$\mathfrak{D} A = R\mathbf{Hom}(A, \mathfrak{D}_X) \in D_{\mathcal{X}}^b(X)$$

is the Verdier dual of A and

$$\mathfrak{D}_X = g^! \mathbb{C} \in D_{\mathcal{X}}^{b,c}(X), \quad g : X \rightarrow pt,$$

is the dualising sheaf [9] (see also [3, 7.18 and Theorem 8.3]). Substituting $A = \mathbb{C}$ into (A.6) one gets

$$\mathfrak{D}_X \boxtimes B \simeq p_2^! B. \quad (\text{A.7})$$

We have

$$\mathfrak{D}_X \boxtimes \mathfrak{D}_Y \simeq \mathfrak{D}_{X \times Y} \quad (\text{A.8})$$

by [3] Corollary 10.26, and

$$\mathfrak{D}^2 \simeq \text{Id}$$

by [9] (see also [3] Theorem 8.10). If $f : X \rightarrow Y$ is a stratified map, then

$$\mathfrak{D} f^* = f^! \mathfrak{D}, \quad \mathfrak{D} f^! = f^* \mathfrak{D}, \quad (\text{A.9})$$

if the stratified map f is proper, or algebraic over \mathbb{C} , or if every fibre of f is compactifiable, then

$$\mathfrak{D} f_* = f_! \mathfrak{D}, \quad \mathfrak{D} f_! = f_* \mathfrak{D} \quad (\text{A.10})$$

by [3] Theorem 10.17.

A.1 Definition (see [1] Proposition 2.1.9). Let $U \subset X$ be an open stratified subspace of a stratified pseudomanifold $X = (X, \mathcal{X}, p)$. Denote by $j : U \hookrightarrow X$ the inclusion. Let $A \in \text{Perv}(U, U \cap \mathcal{X}, p)$. Its *prolongation* $j_{!*}A$ is an object $B \in D_{\mathcal{X}}^{b,c}(X)$ equipped with an isomorphism $j^*B \simeq A$, such that for any stratum $s : S \hookrightarrow X$, that is not contained in U , we have $H^i s^*B = 0$ for $i \geq p(S)$ and $H^i s^!B = 0$ for $i \leq p(S)$. The object B is determined uniquely up to a unique isomorphism. This gives a functor $j_{!*} : \text{Perv}(U, U \cap \mathcal{X}, p) \rightarrow \text{Perv}(X, \mathcal{X}, p)$.

If $k : V \hookrightarrow U$ is another stratified open inclusion, then there is an isomorphism (2.1.7.1) of [1]

$$(j \circ k)_{!*} \xrightarrow{\sim} j_{!*} \circ k_{!*}. \quad (\text{A.11})$$

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